The instability of an annular thread of fluid

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The instability of an annular coating of liquid on a wire or on the inside of a small tube subject to capillary forces at its free surface is discussed. It was found that for given values of s/a, the ratio of the two radii of the annulus, and $S \equiv \rho Ta/\mu^2$, the reciprocal of the square of the Ohnesorge number, there is a disturbance of a certain wavelength $(2\pi a/\lambda)^*$, which grows more rapidly than disturbances of any other wavelength. One would therefore expect the liquid to break up into a regular pattern of drops with spacing given by this wavelength. The dependence of $(2\pi a/\lambda)^*$ on s/a and S has been calculated and is presented graphically. Experimental observations on drop formation on wires and in tubes which agree with the calculations are given.

Introduction

An interesting photograph in the book *Soap Bubbles* by Boys (1959) shows a segment of a spider's web on which droplets of a sticky liquid are regularly spaced. This liquid was originally exuded by the spider as a thin annular film, but capillary forces then caused it to gather into droplets. The high viscosity of the liquid and the presence of the solid core point to the importance of viscous forces. One effect of these is the reduction of the speed of droplet formation. When the viscosity of the liquid is very large and the liquid layer thin, droplets are formed slowly, and inertial effects are negligible in comparison with viscous effects.

Droplet or ripple formation can also occur when a thin liquid coating is applied to a wire or a thread, as, for example, in the coating of wires with molten plastics for insulation or the coating of synthetic fibres with water for lubrication. Again, droplets or ripples can be formed on the inside of a small tube from the thin liquid coating left behind when either the liquid drains from the tube or a large air bubble passes through the tube. Both of the above geometries are found in cylindrical wetted wall columns, used in experimental gas-absorption studies, but unless the flow rate is exceedingly small, the ripples produced will be modified appreciably by the flow.

The mathematical description of this phenomenon follows those given by Rayleigh (1879, 1892), Weber (1931) and Tomotika (1935) for similar problems. In this paper we shall consider a long cylindrical wire (tube) uniformly coated with a liquid of viscosity μ, density ρ, and surface tension T; the inner and outer (outer and inner) radii of the liquid annulus are s and a, respectively. We shall discuss the instability of the liquid annulus when it is subjected to infinitesimal axially symmetric disturbances. It is necessary to consider only a single Fourier component corresponding to a wave along the axis of the annulus, since any infinitesimal disturbance can be represented by an appropriate superposition of such components. The linearized equations of motion contain time only through first derivatives with respect to time, and consequently admit of solutions containing an exponential time factor e^{int}. The boundary conditions on the velocities at the solid surface and the stresses at the free surface are homogeneous, and we therefore have an eigenvalue problem with in and λ, the wavelength, as parameters. If the real part of in is positive, zero, or negative the disturbance initially is amplified, neutrally stable, or damped, respectively. Solution of the characteristic-value equation gives the growth rate of unstable wave-like disturbances as functions of their wavelengths, and if the initial amplitudes of extant minute disturbances are of the same magnitude, droplet formation is controlled by the particular wave which grows most rapidly.

Rayleigh (1879) treated the interactions of inertial and capillary forces acting on a cylinder of an inviscid liquid in the absence of an atmosphere, and has shown that the most rapidly growing disturbance is the one for which $2\pi a/\lambda = 0{\cdot}696$, where a is the initial radius of the liquid cylinder and λ is the wavelength of the disturbance. Rayleigh (1892) also discussed the instability of a cylinder of a viscous liquid without inertia, and found that $2\pi a/\lambda = 0$ for the most rapidly growing mode. These problems are limiting cases of the problem later treated by Weber (1931) in which he considered the break-up of a jet of fluid with viscosity μ, density ρ, and surface tension T. According to his theory, the most rapidly growing mode is given by $2\pi a/\lambda \simeq 0{\cdot}707\{1+(9\mu^2/2\rho Ta)^{\frac{1}{2}}\}^{-\frac{1}{2}}$. Tomotika (1935) considered the interaction of viscous and capillary forces acting on a cylinder of a viscous liquid surrounded by another viscous liquid, and found the controlling mode to be a function of the ratio of the viscosities of the two fluids. For the limiting cases when this ratio is either zero or infinite, the most rapidly growing disturbance is given by $2\pi a/\lambda = 0$, but for a finite value of the ratio, $2\pi a/\lambda$ is non-zero. More recently, Ponstein (1959) has treated jets of fluid which are in a state of solid rotation, and has calculated the decrease in the stability of a solid jet and the increase in the stability of a 'hollow, infinitely thick' jet with an increase in angular velocity. He also considered annular jets with both surfaces free. It is of interest to note that in some cases the non-axially symmetric disturbances are more unstable than axially symmetric ones, whereas in non-rotating jets only axially symmetric disturbances are unstable.

Theory

With the usual notation, the equations of motion and continuity in cylindrical co-ordinates (r, ϕ, z) for axially symmetric motions are

$$\left.\begin{aligned}\partial u/\partial t + u\,\partial u/\partial r + w\,\partial u/\partial z &= -\rho^{-1}\partial p/\partial r + \nu(\partial^2 u/\partial r^2 + r^{-1}\partial u/\partial r - u/r^2 + \partial^2 u/\partial z^2),\\ \partial w/\partial t + u\,\partial w/\partial r + w\,\partial w/\partial z &= -\rho^{-1}\partial p/\partial z + \nu(\partial^2 w/\partial r^2 + r^{-1}\partial w/\partial r + \partial^2 w/\partial z^2),\end{aligned}\right\} \quad (1)$$

and

$$\partial u/\partial r + u/r + \partial w/\partial z = 0. \quad (2)$$

We assume that the motions are proportional to $e^{int+ikz}$ and sufficiently small that squares and products of the velocity components can be neglected. The wavelength λ of the axisymmetric disturbance is related to the wave-number k by the relation $\lambda = 2\pi/k$. The solution to the equations of motion under these conditions has been given by Tomotika, for example. (The notation adopted here is that used by Tomotika.)

The continuity equation is satisfied by a stream function, ψ, such that

$$u = r^{-1}(\partial\psi/\partial z) \quad \text{and} \quad w = -r^{-1}(\partial\psi/\partial r). \tag{3}$$

Eliminating the pressure from the two equations of motion and substituting the above expressions for u and w gives the linearized differential equation for ψ

$$\partial D\psi/\partial t = \nu DD\psi, \tag{4}$$

where

$$D \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{5}$$

The solution to this equation with the assumed t and z variation is

$$\psi = \{A_1 rI_1(kr) + B_1 rK_1(kr) + A_2 rI_1(k_1 r) + B_2 rK_1(k_1 r)\}\, e^{int+ikz}, \tag{6}$$

where

$$k_1^2 = k^2 + in/\nu. \tag{7}$$

$I_n(x)$ and $K_n(x)$ are modified Bessel functions of order n, and A_1, B_1, A_2, B_2 are constants to be determined by the boundary conditions.

The boundary conditions for a wire (tube) coated with liquid are the following.

(1) There is no slip at the solid boundary of the liquid annulus, or

$$u(s) = \left[\frac{1}{r}\frac{\partial\psi}{\partial z}\right]_{r=s} = 0 \quad \text{and} \quad w(s) = -\left[\frac{1}{r}\frac{\partial\psi}{\partial r}\right]_{r=s} = 0. \tag{8}$$

(2) The tangential stress at the free surface of the liquid is zero, or

$$\tau_{RZ}(a) = \mu\left[\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\partial^2\psi}{\partial z^2}\right]_{r=a} = 0. \tag{9}$$

(3) The change in normal stress across the free surface is due to the latter's curvature and the interfacial surface tension. Thus

$$\Delta\tau_{RR}(a) = \Delta\left[-p + 2\mu\frac{\partial u}{\partial r}\right]_{r=a} = T\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{10}$$

Here R_1 and R_2 are the principal radii of curvature of the free surface and p is the hydrostatic pressure. When p and $(1/R_1 + 1/R_2)$ are expressed in terms of the stream function, (10) becomes

$$\left[a\frac{\partial}{\partial r}(a^2D - in\rho a^2/\mu - 2k^2a^2)\,\psi + 2k^2a^2\psi + \frac{\rho Ta/\mu^2}{in\rho a^2/\mu}(1 - k^2a^2)\,k^2a^2\psi\right]_{r=a} = 0. \tag{11}$$

(Of course, if the fluid is inviscid, the boundary conditions of zero tangential velocity at the solid surface and zero shear stress at the free surface must be dropped, consistent with reducing the order of the differential equation for ψ from four to two.)

Application of these boundary conditions leads to a system of four simultaneous linear algebraic equations in the constants A_1, B_1, A_2, B_2 which has a non-trivial solution only if the determinant of the coefficients vanishes. This gives the following relationship between in and k

$$\begin{vmatrix} I_1(ks) & I_1(k_1 s) & K_1(ks) & K_1(k_1 s) \\ ksI_0(ks) & k_1 sI_0(k_1 s) & -ksK_0(ks) & -k_1 sK_0(k_1 s) \\ 2k^2a^2I_1(ka) & (k^2a^2+k_1^2a^2)\,I_1(k_1 a) & 2k^2a^2K_1(ka) & (k^2a^2+k_1^2a^2)\,K_1(k_1 a) \\ F_1 & F_2 & F_3 & F_4 \end{vmatrix} = 0, \tag{12}$$

where

$$\left.\begin{aligned} F_1 &= 2kaI_1'(ka)+\frac{in\rho a^2/\mu}{ka}\,I_0(ka)+\frac{\rho Ta/\mu^2}{in\rho a^2/\mu}\,(k^2a^2-1)\,I_1(ka), \\ F_2 &= 2k_1 aI_1'(k_1 a)+\frac{\rho Ta/\mu^2}{in\rho a^2/\mu}\,(k^2a^2-1)\,I_1(k_1 a), \\ F_3 &= 2kaK_1'(ka)-\frac{in\rho a^2/\mu}{ka}\,K_0(ka)+\frac{\rho Ta/\mu^2}{in\rho a^2/\mu}\,(k^2a^2-1)\,K_1(ka), \\ F_4 &= 2k_1 aK_1'(k_1 a)+\frac{\rho Ta/\mu^2}{in\rho a^2/\mu}\,(k^2a^2-1)\,K_1(k_1 a), \end{aligned}\right\} \tag{13}$$

and where we can write

$$k_1^2a^2 = k^2a^2+in\rho a^2/\mu, \quad ks = ka\,.\,s/a. \tag{14}$$

Equation (12) can thus be regarded as a relation between four dimensionless variables: (i) s/a, a geometrical parameter; (ii) $ka = 2\pi a/\lambda$, a wavelength parameter; (iii) $\rho Ta/\mu^2 \equiv S$, the ratio of the inertial forces to the viscous forces times the ratio of the surface-tension forces to the viscous forces (the Ohnesorge number $= S^{-\frac{1}{2}}$); (iv) $in\rho a^2/u \equiv N$, the growth-rate parameter. If the real part of N is positive, zero, or negative, the disturbance initially is amplified, neutrally stable, or damped, respectively. For amplifying disturbances and for fixed values of s/a and S, we seek that value of $2\pi a/\lambda$ which maximizes the real part of N, and expect the resulting pattern of droplets to be characterized by this value of $2\pi a/\lambda$. Quantities pertinent to the most rapidly growing mode will be denoted by an asterisk.

Equation (12) is a complicated implicit equation for N, which occurs in the argument of some of the Bessel functions, such as $I_1(k_1 s)$, and cannot be solved explicitly for N except in two limiting cases, the case of negligible inertia and the case of negligible viscosity. As the inertial forces become negligible (i.e. $\rho \to 0$), $S \to 0$ and $N \to 0$, but N/S remains a finite function of s/a and $2\pi a/\lambda$. Similarly, as the viscous forces become negligible (i.e. $\mu \to 0$), $S \to \infty$ and $N \to \infty$, but $N/S^{\frac{1}{2}}$ remains a finite function of s/a and $2\pi a/\lambda$. These cases will now be treated in detail, but one should note first that although (12) cannot be solved explicitly for N, it can be solved explicitly for S, a fact which will be useful in discussing the general case.

Case 1. *Negligible inertia*

If we put $N = 0$ directly into (12), the first column becomes identical with the second, and the third with the fourth, giving an indeterminate form. The limiting equation can be obtained using the procedure of Tomotika. The group N is regarded as indefinitely small, and the various functions are expanded in Taylor series of this group. The resulting determinant is simplified, divided by N^2, and the limit taken as N approaches zero. In this way, we obtain the equation

$$\begin{vmatrix} I_1(ks) & ksI_1'(ks) & K_1(ks) & ksK_1'(ks) \\ I_0(ks) & I_0(ks)+ksI_0'(ks) & -K_0(ks) & -K_0(ks)-ksK_0'(ks) \\ I_1(ka) & I_1(ka)+kaI_1'(ka) & K_1(ka) & K_1(ka)+kaK_1'(ka) \\ G_1 & G_2 & G_3 & G_4 \end{vmatrix} = 0, \qquad (15)$$

where

$$\left.\begin{aligned} G_1 &= 2kaI_1'(ka)+\frac{\rho Ta/\mu^2}{in\rho a^2/\mu}(k^2a^2-1)\,I_1(ka), \\ G_2 &= 2\{kaI_1'(ka)+k^2a^2I_1''(ka)-kaI_0(ka)\}+\frac{\rho Ta/\mu^2}{in\rho a^2/\mu}(k^2a^2-1)\,kaI_1'(ka), \\ G_3 &= 2kaK_1'(ka)+\frac{\rho Ta/\mu^2}{in\rho a^2/\mu}(k^2a^2-1)\,K_1(ka), \\ G_4 &= 2\{kaK_1'(ka)+k^2a^2K_1''(ka)+kaK_0(ka)\}+\frac{\rho Ta/\mu^2}{in\rho a^2/\mu}(k^2a^2-1)\,kaK_1'(ka). \end{aligned}\right\} \qquad (16)$$

Using the differentiation and recurrence formulae for Bessel functions, one may express the growth-rate parameter, N, after considerable simplification, as

$$N = S\frac{\frac{1}{2}(k^2a^2-1)\,\Delta_1}{\Delta_2+(k^2a^2+1)\,\Delta_1} = S\Phi_{\text{visc}}(ka, s/a), \qquad (17)$$

where

$$\begin{aligned} \Delta_1 = &-1+2ks\{K_0(ks)\,I_1(ka)+I_0(ks)\,K_1(ka)\}\{K_1(ks)\,I_1(ka)-I_1(ks)\,K_1(ka)\} \\ &+k^2s^2\{K_0(ks)\,I_1(ka)+I_0(ks)\,K_1(ka)\}^2-k^2s^2\{K_1(ks)\,I_1(ka)-I_1(ks)\,K_1(ka)\}^2, \end{aligned} \qquad (18)$$

and

$$\begin{aligned} \Delta_2 = &-(1+k^2s^2)+2ksk^2a^2\{K_0(ka)\,I_1(ks)+I_0(ka)\,K_1(ks)\}\{K_0(ka)\,I_0(ks) \\ &-I_0(ka)\,K_0(ks)\}+k^2s^2k^2a^2\{K_0(ka)\,I_1(ks)+I_0(ka)\,K_1(ks)\}^2 \\ &-k^2s^2k^2a^2\{K_0(ka)\,I_0(ks)-I_0(ka)\,K_0(ks)\}^2. \end{aligned} \qquad (19)$$

The same result may be derived by neglecting originally the inertial terms in the equations of motion and in the boundary conditions. Then the stream function becomes

$$\psi = \{A_1rI_1(kr)+B_1rK_1(kr)+A_2r^2I_0(kr)+B_2r^2K_0(kr)\}\,e^{int+ikz}, \qquad (20)$$

the solution to $DD\psi = 0$.

Case 2. Negligible viscosity

The relation between $N/S^{\frac{1}{2}}$ and s/a and $2\pi a/s$ for an inviscid fluid can be derived from (12) by taking the limit of this equation as N and S approach infinity. However, it is simpler to deal with the inviscid equations of motion directly, and it can be shown that the two methods yield the same result.

Equation (4) becomes $D\psi = 0$ for an inviscid fluid, and has the solution

$$\psi = \{A_1rI_1(kr)+B_1rK_1(kr)\}\,e^{int+ikz}. \qquad (21)$$

The boundary conditions become

$$u(s) = \left[\frac{1}{r}\frac{\partial \psi}{\partial z}\right]_{r=s} = 0. \tag{22}$$

and

$$\left[-a\frac{\partial}{\partial r}(in\rho a^2/\mu)\,\psi + \frac{\rho Ta/\mu^2}{in\rho a^2/\mu}(1-k^2a^2)\,k^2a^2\psi\right]_{r=a} = 0. \tag{23}$$

Again, when ψ is substituted into these boundary conditions, we obtain a set of two homogeneous linear algebraic equations in A_1, B_1 which has a non-trivial solution only if the determinant of the coefficients vanishes. This leads to the relation

$$N^2 = -Ska(1-k^2a^2)\frac{I_1(ks)\,K_1(ka) - K_1(ks)\,I_1(ka)}{I_1(ks)\,K_0(ka) + K_1(ks)\,I_0(ka)} = S\Phi^2_{\text{inviscid}}(ka, s/a). \tag{24}$$

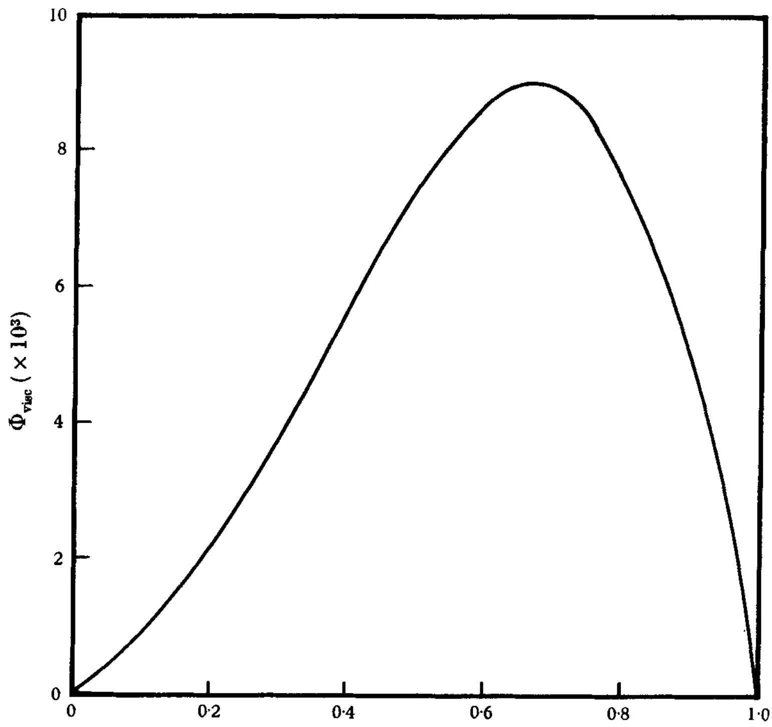

FIGURE 1. Plot of Φ_{visc} as a function of $2\pi a/\lambda$ for $s/a = \frac{1}{2}$.

These equations are valid for $s/a < 1$, i.e. for the geometry of a solid core with a liquid coating. When $s/a > 1$, the geometry is that of a liquid annulus contiguous with the inside of a cylindrical tube. As s/a passes through unity, the system transforms from a very thin liquid layer possessing positive curvature $(1/a)$ at its free surface to one possessing negative curvature $(-1/a)$ at its free surface. This changes the sign of the last term in (11); thus, to obtain N as a function of ka, s/a and S when $s/a > 1$, one needs merely to prefix the right-hand side of (17) or (24) with a minus sign.

Both Φ_{visc} and Φ_{inviscid}, which give the behaviour of N/S when inertial forces are negligible and of $N/S^{\frac{1}{2}}$ when viscous forces are negligible, respectively, are positive real numbers when $0 < 2\pi a/\lambda < 1$, and are negative when $2\pi a/\lambda > 1$. Accordingly, disturbances with wave-numbers between zero and one are unstable and those with wave-numbers greater than one are stable. For a given value of s/a, Φ_{visc} (and Φ_{inviscid} also) has a maximum in the interval $0 \leqslant 2\pi a/\lambda \leqslant 1$. The values $(2\pi a/\lambda)^*$ corresponding to the maxima have been determined by evaluating Φ_{visc} and Φ_{inviscid} as functions of $2\pi a/\lambda$ and using parabolic interpolation about the maximum for several values of s/a as a parameter. Figure 1 shows Φ_{visc} plotted against $2\pi a/\lambda$ for $s/a = \frac{1}{2}$ as a typical plot. The final results are

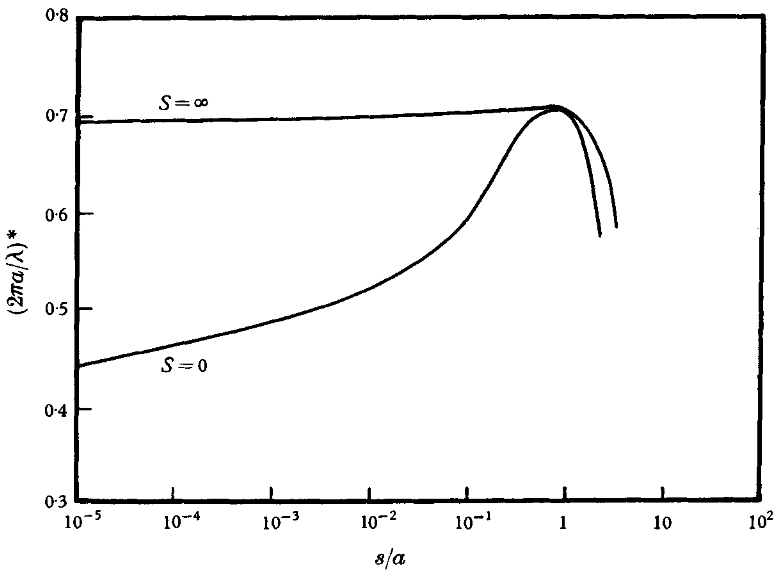

FIGURE 2. Plot of $(2\pi a/\lambda)^*$ as a function of s/a for the two limiting cases, $S = 0$ and $S = \infty$.

given in table 1, where the calculated values of $(2\pi a/\lambda)^*$ and Φ^*_{visc} and $\Phi^*_{\mathrm{inviscid}}$ at the maxima are listed for the several values of s/a. In figure 2 the value of $(2\pi a/\lambda)^*$ so calculated is plotted against s/a for both limiting cases.

In both cases, for a given non-zero value of s/a, there is a non-zero value of $(2\pi a/\lambda)^*$ characteristic of a most rapidly growing disturbance. In accordance with the above, we consequently expect the liquid to form regularly spaced droplets along the wire or tube with finite spacing.

Case 3. *General case*

In general, S is neither zero nor infinite, and the maximum value of N, N^*, and the most rapidly growing disturbance, $(2\pi a/\lambda)^*$, are functions of S as well as s/a. N is proportional to S as $S \to 0$, and proportional to $S^{\frac{1}{2}}$ as $S \to \infty$. We may estimate at what value of S, $\mathscr{S}$ say, the inertial and viscous effects are of comparable magnitude by simultaneously solving the two equations

$$N = S\Phi_{\mathrm{visc}}, \quad N = S^{\frac{1}{2}}\Phi_{\mathrm{inviscid}}. \tag{25}$$

The values of $\mathscr{N}^*$ and $\mathscr{S}^*$ so calculated are also presented in table 1.

In the limit as $s/a \to 0$ (Weber's analysis for a solid jet), $(2\pi a/\lambda)^*$ is a continuously increasing function of S, going from 0 when $S = 0$ to 0·696 when $S = \infty$. The same is true in the limit $s/a \to \infty$ (a hollow jet), the range there being 0 to 0·484. It is therefore reasonable to assume that, for finite S, the curve

s/a	$(2\pi a/\lambda)^*$	Φ^*_{visc}	$(2\pi a/\lambda)^*$	Φ^*_{inviscid}	$\mathscr{N}^*$	$\mathscr{S}^*$
0	0	0·167	0·697	0·343	0·705	4·22
10^{-5}	0·441	0·101	—	—	—	—
10^{-3}	0·486	0·085	—	—	—	—
10^{-2}	0·521	0·072	—	—	—	—
10^{-1}	0·586	0·046	0·697	0·342	2·54	55·3
$\frac{1}{2}$	0·670	0·0090	0·702	0·302	10·1	1130·0
1	0·707	0	0·707	0	∞	∞
1·5	0·672	0·0087	—	—	—	—
2	0·602	0·047	0·674	0·555	6·53	13·9
3	—	—	0·609	0·722	—	—
∞	0	0·500	0·484	0·820	1·34	2·69

TABLE 1. $(2\pi a/\lambda)^*$, Φ^*_{visc} and Φ^*_{inviscid} for various values of s/a.

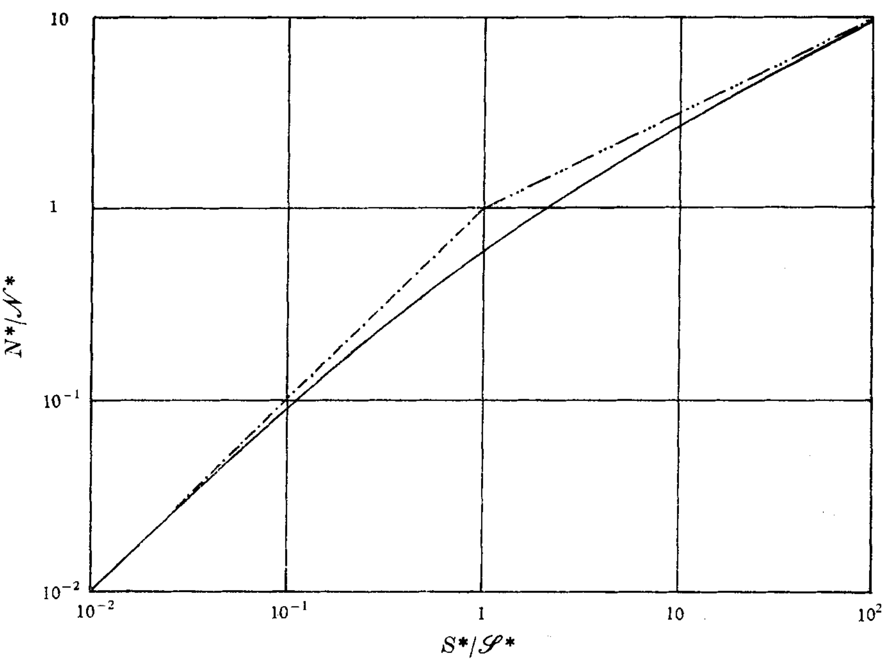

FIGURE 3. Plot of $N^*/\mathscr{N}^*$ as a function of $S^*/\mathscr{S}^*$ for $s/a = \frac{1}{2}$. ——— Equation (12); —·—·— $N^*/\mathscr{N}^* = S^*/\mathscr{S}^*$; —··—·· $N^*/\mathscr{N}^* = (S^*/\mathscr{S}^*)^{\frac{1}{2}}$.

$(2\pi a/\lambda)^*$ versus s/a lies intermediate to the curves for $S = 0$ and $S = \infty$. If this is true, then in the range of greatest practical interest, $0{\cdot}2 \leqslant s/a \leqslant 1{\cdot}5$, the wave length where maximum instability occurs is only weakly dependent upon S as can be seen from figure 2. To avoid a lengthy trial and error calculation the following procedure was adopted. For a given value of s/a, $(2\pi a/\lambda)^*$ was taken as the average for the $S = 0$ and the $S = \infty$ cases. Using this value, and assuming values of N^*, one can calculate S^* explicitly from (12). The results of this calculation are presented in figure 3, were $N^*/\mathscr{N}^*$ is plotted against $S^*/\mathscr{S}^*$ for $s/a = \frac{1}{2}$. With the above assumption, the approximate value of N^* so calculated is a good estimate of the true value because we are interested in the maximum value of N with respect to $2\pi a/\lambda$, i.e. the value of $2\pi a/\lambda$ for which $\partial N/\partial(2\pi a/\lambda) = 0$. Thus a small error in $(2\pi a/\lambda)^*$ will make only a very small error in N^*.

Experiment

Experiments in which drops formed on wires and in tubes were done. In all cases S was sufficiently small for the theory for negligible inertia to be applicable.

Wire experiments

For the wire experiments, a commercial brand of honey (viscosity = 70 P as measured with a Brookfield viscometer, density = $1{\cdot}48\,\mathrm{g/cm^3}$ as measured by weighing a known volume of liquid, and surface tension = 68 dyn/cm as measured with a DuNouy tensiometer) was brushed onto metallic wires with a small paint brush. The two wires used were of radius 0·0127 and 0·0019 cm. Upon application of the liquid, stationary waves were soon observed on the surface of the liquid, and the amplitude of these waves grew until a regular array of droplets was formed. Frequently, as described by Boys, smaller drops would lie between the larger ones. In the wavelength measurements the smaller drops, which appeared to be produced from the thin residual layer connecting the larger ones, were neglected. At later times, droplets were observed to coalesce and a string of droplets would lose its regularity.

After the drops had formed, the wires and arrays of drops were photographed. With the smaller wire, photographs were taken through a microscope with a magnification of 25 ×. The wavelengths were measured from the photograph and the volume of a given drop determined by numerical integration over a wavelength of the square of the radius, also measured from the photograph. Only experiments for which the wavelengths and drop volumes of several consecutive drops along the wire were constant to less than 5 % were accepted.

The results of several experiments, together with the theoretical curve, are shown in figure 4, where $(2\pi a/\lambda)^*$ is plotted against s/a. For comparison, the theoretical results of Rayleigh, Weber and Tomotika for non-supported columns of liquid are also shown. The two theories of Rayleigh give $(2\pi a/\lambda)^* = 0{\cdot}696$ and $(2\pi a/\lambda)^* = 0$ for the inviscid and viscous cases respectively. For Weber's theory, the largest expected value of $(2\pi a/\lambda)^*$ is 0·067 corresponding to the largest value of S used. It should be remembered that the two Rayleigh theories are superseded by Weber's theory, and any comparison should be made with the latter, rather than the former. For Tomotika's theory, the surrounding fluid is taken to be air (viscosity $1{\cdot}8\times10^{-4}$ P) so that $\mu'/\mu = 3{\cdot}9\times10^5$; the value of $(2\pi a/\lambda)^*$ corresponding to this ratio was estimated to be 0·05.

Tube experiments

The tube experiments were performed with a solution of glycerine and water (viscosity = 4·11 P, density $1{\cdot}1\,\mathrm{g/cm^3}$, and surface tension 64 dyn/cm at 23° C) in tubes of an inner radius of 0·0387, 0·0522 and 0·0744 cm. In a given experiment, the capillary tube was filled with the liquid which was then blown out by connecting one end of the tube to a pressure source. In this way, a thin coating of liquid was left on the inner wall of the tube. The speed, U, of the meniscus was determined by measuring the time taken by the meniscus to traverse a known distance. The amount of liquid remaining could then be determined from an empirical relationship between $m = 1 - (a/s)^2$, the fraction of liquid remaining, and the dimensionless group $\mu U/T$ as given by Taylor (1961). Wavelengths were measured directly by measuring the distance occupied by a given number of drops. Again, only measurements for which several consecutive groups of drops were uniform were accepted.

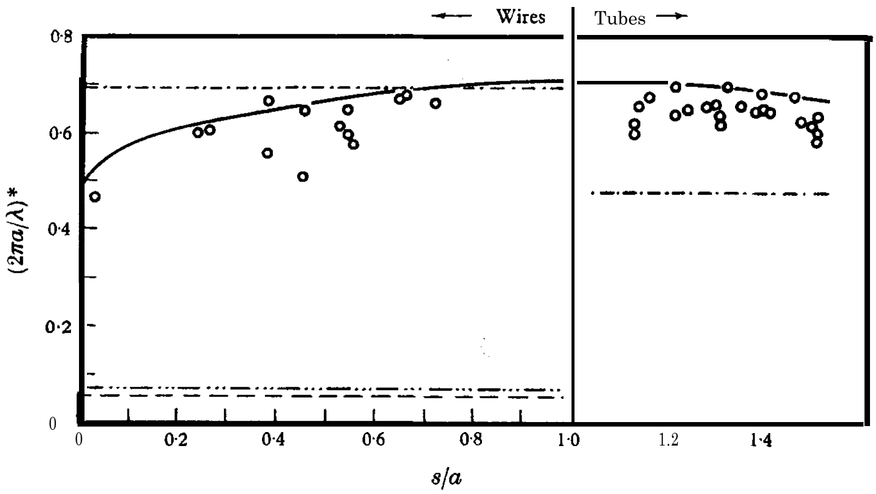

FIGURE 4. Comparison of experimental data with present theory and with theories for unsupported columns of liquid. ——— present theory for negligible inertia; —·—·— Rayleigh's theory for an inviscid column of liquid, $(2\pi a/\lambda)^* = 0{\cdot}696$, and hollow jet, $(2\pi a/\lambda)^* = 0{\cdot}484$; —···— Weber's theory, $(2\pi a/\lambda)^* = 0.067$ for S = 3·8 x 10^{-4}; — Tomotika's theory, $(2\pi a/\lambda)^* = 0.05$ for p'/p = 4 x 106; Rayleigh's theory for a viscous column of liquid, (2na/A)* = 0.

The results are also shown in figure 4. For comparison, the corresponding theoretical results of Rayleigh (i.e. the limit of (2na/h)* in our equations as ~/a becomes infinite) are 0-484 and 0 for the inviscid and viscous cases respectively.

In experiments on the motion of long bubbles in tubes, Bretherton (1961) observed that the rare meniscus had a 'wave-like appearance' which was easily discernible at large film thicknesses. A theory based on the lubrication approximations predicts the existence of this waviness. In view of the instability of the liquid annulus, however, it is possible that the wave is a result of the instability appearing at the rear meniscus, the liquid annulus there having been in existence long enough to afford small disturbances a chance to grow.

As seen from figure 4, the experimental points are consistently slightly low. Possible explanations for this are the non-uniformity of the liquid coating, and the coalescence or partial coalescence of drops. These two effects would tend to make the observed values of $(2\pi a/\lambda)$ lower than expected, for in the first case the importance of the solid core would be lessened making Weber's analysis more applicable, and in the second case an overestimate of λ results in an underestimate of $2\pi a/\lambda$. A third possible explanation is the neglect of higher-order terms in the perturbation analysis. The direction of this effect is not clear, but it must be emphasized that measurements were made on the disturbances only after they had become quite large.

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